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Polyhedral norms on non-separable Banach spaces

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Abstract

We prove the existence of equivalent polyhedral norms on a number of classes of non-separable spaces, the majority of which being of the form $\mathcal{C}(K)$. In particular, we obtain a complete characterization of those trees $\mathcal{T}$, such that $\mathcal{C}_0(\mathcal{T})$ admits an equivalent polyhedral norm.

Keywords: Polyhedral norm; Scattered compact; Tree; Orlicz space

1. Introduction

A Banach space X is called *polyhedral* if the unit ball of each of its finite-dimensional subspaces is a polytope [17]. The simplest example of an infinite-dimensional polyhedral space is c_0 in the natural norm. No infinite-dimensional dual space is polyhedral [19], or even isomorphic to a polyhedral space [7]. Clearly, polyhedrality is an isometric property, i.e. it can be gained or

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lost by the introduction of an equivalent norm. In the study of polyhedral spaces the notion of a *boundary* plays a very important role.

Definition 1. Let X be a Banach space. A subset $B \subset S_{X^*}$ is called a boundary of X if for any $x \in X$ there is $f \in B$ with $f(x) = \|x\|$.

Clearly, for an infinite-dimensional Banach space a boundary must be infinite, i.e. at least *countable*, and it turns out that this is the case in polyhedral spaces (see Theorem 2 below).

By the Krein–Milman theorem, the set $\operatorname{ext} B_{X^*}$ is a boundary. In general, a boundary need not contain all extreme points, but it must contain the w*-exposed points. Recall that a functional $f_0 \in S_{X^*}$ is a w*-exposed point of B_{X^*} if there is a $x_0 \in S_X$ such that $f_0(x_0) = 1 > f(x_0)$ for each $f \in B_{X^*}$, $f \neq f_0$, and that f_0 is called w*-strongly exposed if $\lim \|f_n - f_0\| = 0$ whenever $f_n \in B_{X^*}$ and $\lim f_n(x_0) = 1$. Thus, each boundary B contains the set $B_0 = \text{w}^*\text{-}\operatorname{str}\exp B_{X^*}$ of all the w*-strongly exposed points of B_{X^*}. We summarize some basic properties of polyhedral spaces in the following theorem.

Theorem 2. *(See [5,7].)*

(1) *Let X be a polyhedral Banach space with the density character w. Then the set $B_0 = \text{w}^*\text{-}\operatorname{str}\exp B_{X^*}$ is a boundary of X with $|B_0| = w$. B_0 is a minimal boundary of X, i.e. it is contained in any other boundary. In particular if X is separable then B_0 is countable.*
(2) *If a Banach space X has a countable boundary then X^* is separable and X admits an equivalent polyhedral norm, the boundary of which has* $(*)$ *(see Definition* 4 *below).*
(3) *Every polyhedral space is a c_0-saturated, Asplund space.*

For a simpler proof of the first part of Theorem 2 see [10]. A nice alternative proof of the first part of Theorem 2 is given in [22].

Note that Theorem 2 gives a characterization of the polyhedral spaces in the separable case. Separable polyhedral spaces have many interesting properties, as well as some other characterizations (see [2,5,9–12,18,22]). Unfortunately, not much is known about non-separable polyhedral spaces. Besides general properties of polyhedral spaces described in Theorem 2, we recall that the space $C([1,\alpha])$ of all continuous functions on the segment of ordinals $[1,\alpha]$, where α is arbitrary, admits an equivalent polyhedral norm [6]. Also, M. Jimenez-Sevilla informed us that, under CH, the Kunen compact $\mathcal{K}$ provides an example of a non-separable Asplund space $\mathcal{C}(\mathcal{K})$ that admits no polyhedral renorming. Indeed, in her paper with J.P. Moreno [16, Proposition 4.3] they proved that for any equivalent norm on $\mathcal{C}(\mathcal{K})$, the set of w*-denting points of the dual unit ball is countable. Thus, by Theorem 2, part (1), $\mathcal{C}(\mathcal{K})$ admits no polyhedral norms.

The purpose of this paper is to identify some new classes of non-separable spaces which admit a polyhedral norm. Most of these spaces take the form $\mathcal{C}(K)$, where K is compact. From Theorem 2, part (3), it follows that K must be scattered. In Section 2, we study Talagrand operators, one of our main tools in polyhedral renorming, and prove that $\mathcal{C}(K)$ admits such a renorming if K is a finite product of compact ordinal segments or σ-discrete spaces. An important subclass of scattered compact sets is the class of trees. Due to the fundamental paper [15], the renorming theory of spaces $\mathcal{C}_0(\mathcal{T})$, where $\mathcal{T}$ is a tree, is very rich, and we use some of its results in our paper. In Section 3 we prove one of our main results, Theorem 10, which states that the space $\mathcal{C}_0(\mathcal{T})$ has a polyhedral renorming if and only if it admits a Talagrand operator. Surprisingly enough, according to [15], this happens if and only if $\mathcal{C}_0(\mathcal{T})$ admits a Fréchet differentiable norm, and if and only if $\mathcal{C}_0(\mathcal{T})$ admits a LUR norm. By using Theorem 10, we obtain ZFC examples of scattered compacts K with the property that $\mathcal{C}(K)$ does not admit a polyhedral renorming. In Section 4 we establish a sufficient condition for a Banach space with *uncountable* unconditional basis to have a polyhedral renorming. Then we apply this result to non-separable Orlicz spaces.

In the remainder of this introduction, we prove two results used throughout the paper, and end with a discussion of a couple of open problems. Let us first note that the canonical supremum norm in $\mathcal{C}(K)$ is not polyhedral if K is infinite. This follows from two well-known facts: $\mathcal{C}(K)$ contains an isometric copy of $(c, \|\cdot\|_\infty)$ and $(c, \|\cdot\|_\infty)$ is not polyhedral. The following necessity condition is a generalization of the latter fact (take $T_n : (c, \|\cdot\|_\infty) \to (c, \|\cdot\|_\infty)$, $n = 1, 2, \ldots$, defined by $(T_n y)_i = y_i$ if $i \leqslant n$, and zero, otherwise).

Lemma 3. *Let Y be a polyhedral Banach space. Suppose that there exists a family $\mathcal{L}$ of linear maps $T : Y \to Y$ such that*

$$\|y\| = \sup\bigl\{\|Ty\|:\ T \in \mathcal{L}\bigr\} \quad \textit{for all } y \in Y. \tag{1.1}$$

Then Y^* is the norm closed linear span of $\bigcup_{T\in\mathcal{L}} T^*Y^*$.

Proof. From (1.1), it follows that $\|T\| \leqslant 1$ for any $T \in \mathcal{L}$. Put

$$B = \mathrm{w}^*\text{-cl} \bigcup_{T\in\mathcal{L}} T^*(B_{Y^*}).$$

By using (1.1) and the fact that B is w*-compact it is easy to see that B contains a boundary. Now by Theorem 2, part 1, $\mathrm{w}^*\text{-str}\exp B_{Y^*} \subset B$. By the definition of w*-strongly exposed points, we have $\mathrm{w}^*\text{-str}\exp B_{Y^*} \subset \mathrm{cl} \bigcup_{T\in\mathcal{L}} T^*(B_{Y^*})$. Since $B_{Y^*} = \mathrm{cl}\,\mathrm{co}\,\mathrm{w}^*\text{-str}\exp B_{Y^*}$ by [10,22], the conclusion follows. □

Next, we present our main tool used in constructing polyhedral renormings.

Definition 4. We say that a set $B \subset B_{X^*}$ has property $(*)$ if, given any w*-limit point f_0 of B (i.e. any w*-neighborhood of f_0 contains infinitely many points of B), we have $f_0(x) < 1$ whenever $x \in S_X$. If a Banach space in some norm has a boundary with $(*)$, we simply say that the norm itself has $(*)$.

It is not difficult to see that if B has $(*)$ and is 1-norming then B contains a boundary. The next proposition demonstrates the relevance of property $(*)$. We give a proof for the sake of completeness.

Proposition 5. *(See [12].) Assume that a Banach space X admits a* 1*-norming subset $B \subset B_{X^*}$ with $(*)$. Then X is a polyhedral space.*

Proof. Let $E \subset X$ be finite-dimensional. Evidently $\|x\| = \sup_{f\in B} f(x)$. Assume, for a contradiction, that whenever $F \subset B$ is finite, there is $x \in E$ such that $\max_{f\in F} f(x) < \|x\|$. In this way we obtain a sequence $\{f_i\}_{i=1}^\infty \subset B$ of distinct points and $\{x_i\}_{i=1}^\infty \subset S_E$ such that $f_i(x_i) \to 1$. By compactness and by considering subsequences if necessary, we can find $x \in S_E$ such that $\|x_i - x\| \to 0$ and a w*-accumulation point $f \in B_{X^*}$ of $\{f_i\}_{i=1}^\infty$. It is clear that $f(x) = 1$, contradicting property $(*)$ of B. □

To finish, we present two open problems. It was proved in [4,5] that a separable Banach space is isomorphically polyhedral if and only if it has a boundary with $(*)$, with respect to some equivalent norm. Given that all the polyhedral norms constructed in this paper have boundaries with $(*)$, it is natural to ask whether this equivalence holds for non-separable Banach spaces also.

We mention another property, related to polyhedrality, which has attracted attention in, for example, the theory of smooth renormings. A norm $\|\cdot\|$ on X is said to depend *locally on finitely many coordinates* if, given any non-zero $x \in X$, there exists an open set U containing x, a function Φ and finitely many functionals $f_1, \ldots, f_n \in X^*$ such that $\|y\| = \Phi(f_1(x), \ldots, f_n(x))$ whenever $y \in U$. By modifying the proof of Proposition 5, it is clear that if $\|\cdot\|$ has $(*)$ then it depends locally on finitely many coordinates (with $\Phi = \max$). In [9], it was shown that a separable space X is isomorphically polyhedral if and only if it has an equivalent norm that depends locally on finitely many coordinates. Again, we can ask whether this result holds in the non-separable case.

2. Scattered compact spaces

By Theorem 2, part (3), if, for a compact space K, the space $\mathcal{C}(K)$ admits an equivalent polyhedral norm, then $\mathcal{C}(K)$ is an Asplund space. It is well known that $\mathcal{C}(K)$ is Asplund if and only if K is scattered, so when investigating the problem of polyhedral renormings of $\mathcal{C}(K)$ spaces, we only need to consider scattered K.

In this section, we develop some general techniques and apply them to two important classes of scattered, compact spaces. First, we show that if K is a compact ordinal segment then $\mathcal{C}(K)$ admits a polyhedral renorming, thus providing another proof of the result in [6] stated in the introduction. Second, we show that if K is a σ-discrete space then the same conclusion holds. We go on to prove that the same is also true if K is a finite product of spaces of either class. While we do encounter tree spaces in this section, a fuller treatment is deferred until the following section.

We begin stating a natural generalization of Haydon's definition of Talagrand operators [14], which have been used to considerable effect in the theory of smooth renormings on $\mathcal{C}(K)$ spaces. According to [14], given the space $\mathcal{C}_0(L)$, with L locally compact, a linear bounded operator $T : \mathcal{C}_0(L) \to c_0(L \times M)$ is a Talagrand operator if, for any $x \in \mathcal{C}_0(L)$, there is a pair $(t, m) \in L \times M$ with $x(t) = \|x\|_\infty$ and $(Tx)(t, m) \neq 0$.

Definition 6. Let X be a Banach space and M be a non-empty set. A linear, bounded operator $T : X \to c_0(S_{X^*} \times M)$ is called a Talagrand operator if, for any $x \in X$, there is a pair $(f, m) \in S_{X^*} \times M$ with $f(x) = \|x\|$ and $(Tx)(f, m) \neq 0$.

Clearly, if an operator satisfies Haydon's definition then it is also a Talagrand operator in our sense. Some of the general theory of these operators is developed in [21]. The canonical example of a Talagrand operator T is defined on $\mathcal{C}([0, \alpha])$, where $[0, \alpha]$ is a compact ordinal segment. We set (see [14])

$$(Tx)(\xi) = \begin{cases} x(\xi) - x(\xi + 1), & \xi < \alpha, \\ x(\alpha), & \xi = \alpha. \end{cases}$$

Note that a similar construction was used in [6]. Here, the set M is a singleton so we can safely ignore it. If $x \in \mathcal{C}([0, \alpha])$ is non-zero then $(Tx)(\xi) \neq 0$, where ξ is *maximal*, subject to the condition that $\|x\|_\infty = |x(\xi)|$. To see that T maps into $c_0([0, \alpha])$, observe that by the

Stone–Weierstrass theorem, $\mathcal{C}([0,\alpha])$ is the closed linear span of the set of indicator functions $\{\mathbf{1}_{[0,\xi]}\colon\ \xi \leqslant \alpha\}$, so it is enough to check that T maps the set of these indicator functions into $c_0([0,\alpha])$. This is clear however, as we can see by inspection that the support of $T\mathbf{1}_{[0,\xi]}$ is the singleton $\{\xi\}$.

Proposition 7. *Assume that a Banach space X admits a Talagrand operator. Then, for any $\varepsilon > 0$, X admits an ε-equivalent polyhedral norm with $(*)$.*

Proof. We shall assume that $\|T\| = \varepsilon$. For any $f \in S_{X^*}$ set

$$A_f = \{T^*\delta_{(f,m)}\colon\ m \in M\}$$

where $\delta_{(f,m)}$ is the evaluation functional at (f,m). Then put $A = \bigcup_{f\in S_{X^*}} A_f$ and

$$B = \bigcup_{f\in S_{X^*}} f \pm A_f.$$

Since T acts into $c_0(S_{X^*} \times M)$, it is easy to see that the only w*-limit point of the set A is the origin.

Introduce in X the following norm

$$|||x||| = \sup_{g\in B}\bigl\{g(x)\bigr\}, \quad x \in X.$$

From the definition of a Talagrand operator, it follows that for any non-zero $x \in X$, we have

$$|||x||| > \|x\|. \tag{2.2}$$

On the other hand, we have $|||x||| \leqslant (1+\varepsilon)\|x\|$ for any $x \in X$. Therefore the norm $|||\cdot|||$ is ε-equivalent to the original one.

We need to check that B has $(*)$. Let g be a w*-limit point of B. Since the only w*-limit point of A is the origin, it follows that $g \in B_{X^*}$. Assume that there is $x \in X$, $|||x||| = 1$, with $g(x) = 1$. We have

$$\|x\| \geqslant g(x) = 1 = |||x|||$$

contradicting (2.2). Thus B has $(*)$ as required. □

Corollary 8. *(See [6].) If $\varepsilon > 0$ and α is any ordinal then $\mathcal{C}[0,\alpha]$ admits an ε-equivalent polyhedral norm with $(*)$.*

Recall that a tree $(\mathcal{T}, \preccurlyeq)$ is a partially ordered set, such that given any $t \in \mathcal{T}$, the set of predecessors $\{s \in \mathcal{T}\colon\ s \preccurlyeq t\}$ is well ordered. Further definitions relating to trees will be given in the next section. There, we shall see that trees give rise to scattered, locally compact spaces.

Example 9. We say that a tree $\mathcal{T}$ is *special* if it is a countable union of *antichains*; that is to say, $\mathcal{T} = \bigcup_{i=0}^{\infty} A_i$, where distinct elements of any given A_i are incomparable. It is evident that any tree of countable height is special.

For any special tree $\mathcal{T}$, there is a Talagrand operator $T : \mathcal{C}_0(\mathcal{T}) \to c_0(\mathcal{T})$. To construct T, let $\mathcal{T} = \bigcup_{i=0}^{\infty} A_i$, where the sets A_i are pairwise disjoint antichains. Define $T : \mathcal{C}_0(\mathcal{T}) \to l_\infty(\mathcal{T})$ by

$$(Tx)(t) = 2^{-i} x(t), \quad \text{whenever } t \in A_i.$$

If $x \in \mathcal{C}_0(\mathcal{T})$ is non-zero then given any $t \in \mathcal{T}$ such that $|x(t)| = \|x\|$, we have $(Tx)(t) \neq 0$. As with the ordinal example, we use the Stone–Weierstrass theorem to show that T maps into $c_0(\mathcal{T})$. Indeed, $\mathcal{C}_0(\mathcal{T})$ is the closed linear span of the set of indicator functions $\{\mathbf{1}_{(0,t]}\colon t \in \mathcal{T}\}$, where $(0, t]$ denotes the set of all predecessors of t in $\mathcal{T}$. Thus, to prove that T acts into $c_0(\mathcal{T})$, it is sufficient to check that $T\mathbf{1}_{(0,t]} \in c_0(\mathcal{T})$ for every t. However, this is clear because if $\varepsilon > 0$ and $2^{-n} < \varepsilon$ then, by the antichain property, there are at most n elements $s \in (0, t]$ satisfying $T\mathbf{1}_{(0,t]}(s) \geqslant \varepsilon$.

Subtler examples of Talagrand operators on $\mathcal{C}_0(\mathcal{T})$ can be found in [15].

A complete characterization of polyhedral renormings on tree spaces is given in the next theorem.

Theorem 10. *If $\mathcal{T}$ is a tree then the following are equivalent.*

(1) *$\mathcal{C}_0(\mathcal{T})$ admits a polyhedral renorming*;
(2) *for any $\varepsilon > 0$, $\mathcal{C}_0(\mathcal{T})$ admits an ε-equivalent polyhedral renorming with* $(*)$;
(3) *$\mathcal{C}_0(\mathcal{T})$ admits a Talagrand operator.*

This characterization turns out to be exactly the same as that of equivalent Fréchet norms on $\mathcal{C}_0(\mathcal{T})$ although there is no indication that the same is true of $\mathcal{C}(K)$, for general compact K. In fact, we do not know any examples of polyhedral Banach spaces which lack Fréchet renormings.

After Proposition 7, to complete the proof of this theorem, it is enough to prove that if $\mathcal{C}_0(\mathcal{T})$ admits no Talagrand operators, then $\mathcal{C}_0(\mathcal{T})$ has no polyhedral renormings. We postpone the proof of this fact to the next section (see Theorem 14).

Now we turn to the class of σ-discrete compact spaces. A compact set K is *σ-discrete* if it can be written as a countable union of sets $\{D_i\}_{i=1}^{\infty}$, each of which is discrete in its relative topology. For example, if K is compact and the derived set $K^{(\omega_1)}$ of order ω_1 is empty, then $K^{(\beta)}$ is empty for some $\beta < \omega_1$ and so $K = \bigcup_{\alpha<\beta}(K^{(\alpha)} \setminus K^{(\alpha+1)})$ is σ-discrete. The idea behind the proof of the following theorem is based on a result in [13].

Theorem 11. *Let K be a σ-discrete compact set. Then for every $\varepsilon > 0$, $\mathcal{C}(K)$ admits an ε-equivalent polyhedral norm with* $(*)$.

Proof. Let $K = \bigcup_{i=1}^{\infty} D_i$, where each D_i is discrete, and let $\varepsilon > 0$. Define $I_t = \{i \in \mathbb{N}\colon t \in \operatorname{cl} D_i\}$ and

$$\psi(t) = 1 + \varepsilon \sum_{i \in I_t} 2^{-i}.$$

We specify a norm $\|\cdot\|$ on $\mathcal{C}(K)$ by setting

$$\|x\| = \sup_{t \in K} \bigl\{\psi(t)\bigl|x(t)\bigr|\bigr\}.$$

It is clear that $\|\cdot\|_\infty \leqslant \|\cdot\| \leqslant (1+\varepsilon)\|\cdot\|_\infty$ and the set $B = \{\pm\psi(t)\delta_t\colon t \in K\}$ is 1-norming for $(\mathcal{C}(K), \|\cdot\|)$.

Now we show that B has $(*)$. Let f be a w*-limit point of B. Then $f = \alpha\delta_t$ for some $t \in K$ and $\alpha \in \mathbb{R}$. We claim $|\alpha| < \psi(t)$, thus giving $\|f\| < 1$.

We begin by picking $n \in \mathbb{N}$ such that $t \in D_n$. As D_n is discrete, we can find an open set V satisfying $(\operatorname{cl} D_n) \cap V = \{t\}$. If $s \in V$ and $s \neq t$ then $n \in I_t \setminus I_s$.

Let us define $J = I_t \cup \{i \in \mathbb{N}\colon i > n+1\}$ and consider the open set

$$U = K \setminus \bigcup\{\operatorname{cl} D_i\colon i \in \mathbb{N} \setminus J\}.$$

By the definition of J, it is clear that $t \in U$ and $I_s \subset J$ whenever $s \in U$. Moreover, the way we choose V gives that $n \notin I_s$ for each $s \in (U \cap V) \setminus \{t\}$ and we have:

$$I_s \setminus I_t \subset J \setminus I_t \subset \{i \in \mathbb{N}\colon i > n+1\}.$$

Thus, for such an s we obtain the following inequality

$$\psi(t) - \psi(s) = \varepsilon \sum_{i \in I_t \setminus I_s} 2^{-i} - \varepsilon \sum_{i \in I_s \setminus I_t} 2^{-i} \geqslant \varepsilon 2^{-n} - \varepsilon \sum_{i > n+1} 2^{-i} = \varepsilon 2^{-n-1} > 0.$$

Now we can find a net $\{s_\lambda\} \subset (U \cap V) \setminus \{t\}$ such that $\lim s_\lambda = t$ and $\lim \psi(s_\lambda) = |\alpha|$. From this, it is evident that $\psi(t) - |\alpha| \geqslant \varepsilon 2^{-n-1} > 0$. $\square$

We remark that Proposition 7 and Theorem 11 apply to incomparable classes of spaces. The Ciesielski–Pol compact space K, introduced in [1], has the property that there is no injective linear map $T : \mathcal{C}(K) \to c_0(\Gamma)$, for any set Γ. Moreover, $K^{(3)}$ is empty. As Talagrand operators are evidently injective, this means that $\mathcal{C}(K)$ cannot admit such an operator. On the other hand, the compact $[0, \omega_1]$ is not σ-discrete.

Our last results of this section concern renorming injective tensor products. For convenience, we review some basic facts about these products. Given Banach spaces X and Y, the injective product $X \otimes_\varepsilon Y$ is the completion of the algebraic product $X \otimes Y$ with respect to the norm

$$\left\|\sum x_i \otimes y_i\right\| = \sup\left\{\sum f(x_i)g(y_i)\colon f \in B_{X^*}, g \in B_{Y^*}\right\}.$$

If K is compact then $\mathcal{C}(K) \otimes_\varepsilon Y$ identifies with the space $\mathcal{C}(K; Y)$ of continuous Y-valued functions on K. In particular, given compact spaces K_1 and K_2, we have $\mathcal{C}(K_1) \otimes_\varepsilon \mathcal{C}(K_2) \equiv \mathcal{C}(K_1; \mathcal{C}(K_2)) \equiv \mathcal{C}(K_1 \times K_2)$.

If $f \in X^*$ and I_Y is the identity operator on Y then we define $f^Y = f \otimes I_Y$ on $X \otimes Y$ by $f^Y(\sum x_i \otimes y_i) = \sum f(x_i) y_i$. We have $\|f^Y\| = \|f\|$ and extend f^Y to the completion. Similarly we define g^X for $g \in Y^*$. Note that $g \circ f^Y = f \otimes g = f \circ g^X$ whenever $f \in X^*$ and $g \in Y^*$, and $\|u\| = \sup_{f \in A} \|f^Y(u)\| = \sup_{(f,g) \in A \times B} (f \otimes g)(u) = \sup_{g \in B} \|g^X(u)\|$ for all $u \in X \otimes_\varepsilon Y$ and all 1-norming subsets $A \subset B_{X^*}$, $B \subset B_{Y^*}$.

We shall denote the strong operator topology by *SOT*. Observe that the map $f \mapsto f^Y$ is w*-to-*SOT* continuous on bounded subsets of X^*. Indeed, if $\{f_\lambda\}$ is a bounded net converging to f in the w*-topology then, by inspection, we have $f_\lambda^Y(\sum x_i \otimes y_i) \to f^Y(\sum x_i \otimes y_i)$ in norm; the boundedness of the net allows us to extend this convergence to points in the completion. As an immediate corollary, we have $\|u\| = \|f^Y(u)\| = \|g^X(u)\|$ for some $f \in S_{X^*}$ and $g \in S_{Y^*}$.

Amongst other things, the next result is a generalization of Proposition 5.

Theorem 12. *Suppose that X admits a boundary with $(*)$ and Y admits a polyhedral renorming. Then $X \otimes_\varepsilon Y$ admits a polyhedral renorming. Moreover, if both X and Y admit boundaries with $(*)$ then so does $X \otimes_\varepsilon Y$.*

Proof. Let $\|\cdot\|$ denote the norms on X, Y and $X \otimes_\varepsilon Y$, where $\|\cdot\|$ is polyhedral on X and Y. Assume that B is a boundary of X with $(*)$. Since B is 1-norming,

$$\|u\| = \sup_{f \in B} \{ \| f^Y(u) \| \}$$

for $u \in X \otimes_\varepsilon Y$. We claim that the natural tensor norm is polyhedral. First of all, we show that if $\|u\| = 1$ then there is an open neighborhood U of u and a finite set $F \subset B$, such that $\|v\| = \max_{f \in B}\{\|f^Y(v)\|\}$ whenever $v \in U$. Indeed, if the negation holds then we can find a sequence $\{u_i\}_{i=1}^\infty$ converging to $u \in S_{X \otimes_\varepsilon Y}$ in norm, together with a sequence of distinct elements $\{f_i\}_{i=1}^\infty \subset B$ such that $\|u_i\| - \|f_i^Y(u_i)\| \to 0$. It follows that $\|f_i^Y(u)\| \to 1$. If f is a w*-accumulation point of $\{f_i\}$ then, by the remarks above, f^Y is a *SOT*-accumulation point of $\{f_i^Y\}$, whence $\|f^Y(u)\| = 1$. If we take $g \in S_{Y^*}$ such that $g(f^Y(u)) = 1$, we have $1 = f(g^X(u)) \leqslant \|g^X(u)\| \leqslant \|u\| = 1$, meaning B does not have $(*)$. Therefore, locally about points on the sphere, $\|\cdot\|$ depends on only finitely many elements of B.

Let $E \subset X \otimes_\varepsilon Y$ be a finite-dimensional subspace. Given $u \in S_E$, we can take U and F as above. The sum $H = \sum_{f \in F} f^Y(E)$ is a finite-dimensional subspace of Y and thus there exists a finite set $G \subset S_{Y^*}$ such that $\|y\| = \max_{g \in G}\{g(y)\}$ whenever $y \in H$. Consequently,

$$\|v\| = \max_{(f,g) \in F \times G} \{ (f \otimes g)(v) \}$$

whenever $v \in U \cap E$. By a simple compactness argument applied to S_E, it follows that B_E is a polytope.

We move on to the case where Y also has a boundary B' with $(*)$. This time, the natural tensor norm can be written as

$$\|u\| = \sup_{(f,g) \in B \times B'} \{ (f \otimes g)(u) \}.$$

We prove that $D = \{f \otimes g\colon (f, g) \in B \times B'\}$, which is evidently a 1-norming subset of $B_{X \otimes_\varepsilon Y}$, has $(*)$. To this end, suppose that h is a w*-accumulation point of D and, for a contradiction, let us assume that $h(u) = 1$ for some $u \in S_{X \otimes_\varepsilon Y}$. For $n \in \mathbb{N}$, consider the set

$$I_n = \{ f \in B\colon \left| (f \otimes g)(u) - 1 \right| < n^{-1} \text{ for some } g \in B' \}.$$

There are 2 cases: (a) I_n is infinite for every n or (b) there is an n_0 such that I_{n_0} is finite. In case (a), we can extract sequences $\{f_i\}_{i=1}^\infty \subset B$, $\{g_i\}_{i=1}^\infty \subset B'$ such that $(f_i \otimes g_i)(u) \to 1$ and $f_i \neq f_j$ whenever $i \neq j$. If g is a w*-accumulation point of the sequence $\{g_i\}$, then g^X is a *SOT*-accumulation point of $\{g_i^X\}$ and thus, by extracting a subsequence if necessary, we can assume that $(f_i \otimes g)(u) \to 1$. Finally, if f is a w*-accumulation point of $\{f_i\}$ then $f(g^X(u)) = (f \otimes g)(u) = 1$, which contradicts the fact that B has $(*)$, as the f_i are distinct and $1 \leqslant \|g^X(u)\| \leqslant \|u\| = 1$.

If (b) holds instead, it must be that J_n is infinite for all n, where

$$J_n = \{g \in B' \colon \left|(f \otimes g)(u) - 1\right| < n^{-1} \text{ for some } f \in B\}.$$

Indeed, otherwise there exists n_0 such that I_{n_0} and J_{n_0} are both finite, whence there is a neighborhood of h that only meets finitely many elements of D. Hence we can simply exchange the roles of f_i and g_i in the argument above to contradict the fact that B' has $(*)$. $\square$

The following is an immediate corollary of Proposition 7, Theorems 11 and 12.

Corollary 13. *Let $K_1, \dots, K_n$ be compact spaces. If, for every $\varepsilon > 0$ and $i \leqslant n$, $\mathcal{C}(K_i)$ admits an ε-equivalent norm with $(*)$, then so does $\mathcal{C}(\prod_{i=1}^n K_i)$. In particular, if K_i is either an ordinal segment or σ-discrete, then for every $\varepsilon > 0$, $\mathcal{C}(\prod_{i=1}^n K_i)$ admits an ε-equivalent norm with $(*)$.*

As shown in [21], not every finite, Cartesian product of ordinals K admits a Talagrand operator on $\mathcal{C}(K)$, so we cannot make a direct application of Proposition 7 in this context. For such finite products of ordinals K, it is also the case that $\mathcal{C}(K)$ admits both LUR [20] and Fréchet renormings [14].

3. The proof of Theorem 10

The aim of this section is to complete the proof of Theorem 10. As mentioned before, we need to prove the following result.

Theorem 14. *Suppose that $\mathcal{T}$ is a tree such that $\mathcal{C}_0(\mathcal{T})$ admits no Talagrand operators, then $\mathcal{C}_0(\mathcal{T})$ has no polyhedral renormings.*

First, we introduce some definitions and theory concerning trees, including Haydon's characterization of trees $\mathcal{T}$ such that $\mathcal{C}_0(\mathcal{T})$ admits a Talagrand operator [15, Theorem 8.1] (see Theorem 22 below). After that we show in Lemma 23 that if $\mathcal{C}_0(\mathcal{T})$ does not admit such an operator, then for any equivalent norm $\|\cdot\|$ on $\mathcal{C}_0(\mathcal{T})$ there exists a subspace Y of $(\mathcal{C}_0(\mathcal{T}), \|\cdot\|)$, and a sequence of linear operators $\{T_n : Y \to Y \colon n \in \mathbb{N}\}$ such that $\|y\| = \sup\{\|T_n y\| \colon n \in \mathbb{N}\}$ for all $y \in Y$, and Y^* is not the norm closed linear span of $\bigcup_{n \in \mathbb{N}} T_n^* Y^*$. Thus, by Lemma 3, Y and $(\mathcal{C}_0(\mathcal{T}), \|\cdot\|)$ are not polyhedral.

Let $\mathcal{T}$ be a tree. For convenience, we fix an element 0, not in $\mathcal{T}$, such that $0 \prec t$ for all $t \in \mathcal{T}$. Given $s \in \mathcal{T} \cup \{0\}$ and $t \in \mathcal{T}$ with $s \prec t$, we set $(s, t] = \{\xi \colon s \prec \xi \preccurlyeq t\}$ and $[t, \infty) = \{u \in \mathcal{T} \colon u \succcurlyeq t\}$. The meanings of (s, t), $[s, t)$ and (t, ∞) should be clear. The *interval topology* on $\mathcal{T}$ takes as a basis all sets of the form $(s, t]$ as above. This topology is locally compact because each basis element $(s, t]$ is compact, and it is scattered since any minimal element of a given non-empty subset $A \subset \mathcal{T}$ is isolated in A. In order for the topology to be Hausdorff also, we require that every non-empty, totally ordered subset of $\mathcal{T}$ has at most one minimal upper bound. Henceforth we assume that all our trees have this property. Note that the Hausdorff property also allows us to define a meet $\wedge$ on pairs of elements of $\mathcal{T}$ that have a common predecessor;

if $(0,s] \cap (0,t]$ is non-empty then we set $s \wedge t = \sup(0,s] \cap (0,t]$. For the rest of this section, we fix an equivalent norm $\|\cdot\|$ on $\mathcal{C}_0(\mathcal{T})$, satisfying $\|\cdot\| \leqslant \|\cdot\|_\infty \leqslant M\|\cdot\|$.

Central to the proof we seek is the notion of increasing functions on trees. The map $\rho : \mathcal{T} \to \mathbb{R}$ is *increasing* if $\rho(s) \leqslant \rho(t)$ whenever $s \preccurlyeq t$.

Definition 15. Given a function $f \in \mathcal{C}_0(\mathcal{T})$, $a \in \mathbb{R}$ and $t, u \in \mathcal{T}$ with $t \preccurlyeq u$, we define

$$\mu(f,a,t,u) = \inf\bigl\{\|f + a\mathbf{1}_{(t,u]} + \varphi\|: \operatorname{supp}\varphi \subset (u,\infty)\bigr\}.$$

In this way, we extend slightly the definition of Haydon's so-called *μ-functions* (see [15, Section 3] and elsewhere throughout the paper). In Haydon's definitions, f is restricted to a certain subset of $\mathcal{C}_0(\mathcal{T})$ which depends on t, whereas here, f is arbitrary and thus there is a need to specify t explicitly. For convenience, let us also define

$$\mu(f,t) = \mu(f,a,t,t) = \inf\bigl\{\|f + \varphi\|: \operatorname{supp}\varphi \in (t,\infty)\bigr\}.$$

We see immediately that for fixed f and a, the functions

$$u \mapsto \mu(f,a,t,u) \quad \text{and} \quad t \mapsto \mu(f\mathbf{1}_{\mathcal{T}\setminus(t,\infty)}, t)$$

are increasing on the domains $[t,\infty)$ and $\mathcal{T}$, respectively. By elementary reasoning, it is apparent that the functions

$$f \mapsto \mu(f,a,t,u) \quad \text{and} \quad a \mapsto \mu(f,a,t,u)$$

are 1-Lipschitz, which is a fact to be exploited in several approximation arguments later on. In this section, we pay attention to the first type of increasing function above. In particular, we describe two situations in which the infimum in the definition of $\mu(f,a,t,u)$ is attained. The following material is a mild generalization of [15, Lemma 3.1], [15, Proposition 3.4] and associated remarks. Proofs are provided for convenience. If $t \in \mathcal{T}$ then we write t^+ for the set of immediate successors of t.

Definition 16. Let $\rho : \mathcal{T} \to \mathbb{R}$ be an increasing function. Then $t \in \mathcal{T}$ is a *bad point* for ρ if there is a sequence of distinct points $\{u_i\}_{i=1}^\infty \subset t^+$ such that $\lim \rho(u_i) = \rho(t)$.

Observe that if $u \succcurlyeq t$ is a bad point for $\mu(f,a,t,\cdot)$ then

$$\mu(f,a,t,u) = \|f + a\mathbf{1}_{(t,u]}\|.$$

Indeed, we take a sequence of distinct points $\{v_i\}_{i=1}^\infty \subset u^+$ and functions $\{\varphi_i\}_{i=1}^\infty$, where φ_i is supported on (v_i,∞), such that

$$\|f + a\mathbf{1}_{(t,v_i]} + \varphi_i\| \leqslant \mu(f,a,t,v_i) + 2^{-i}.$$

Since $(f + a\mathbf{1}_{(t,v_i]} + \varphi_i)$ converges pointwise to $f + a\mathbf{1}_{(t,u]}$, we have weak convergence because $\mathcal{T}$ is scattered, and therefore $\|f + a\mathbf{1}_{(t,u]}\| \leqslant \mu(f,a,t,u)$ as required.

We move on to the second example of infimum attainment.

Definition 17. A subset $E \subset \mathcal{T}$ is said to be *ever-branching* if, given $t \in E$, there exist incomparable elements $u, v \in E$ such that $t \prec u, v$.

The simplest type of ever-branching subset is a dyadic tree of height ω. The significance of ever-branching subsets for the μ-functions is explained by the following result, which is a cosmetic generalization of [15, Proposition 3.4].

Lemma 18 *(Haydon). Let $E \subset \mathcal{T}$ be an ever-branching subset and suppose that $t \in \mathcal{T}$ and $u \in E$, with $t \preccurlyeq u$. Then there exists a function $\psi \in \mathcal{C}_0(\mathcal{T})$, $\|\psi\|_\infty = 1$, supported on $(t, u] \cup (u, \infty)$ and satisfying*:

(1) *$\psi(s) = 1$ for $s \in (t, u]$;*
(2) *if $f \in \mathcal{C}_0(\mathcal{T})$ and $\mu(f, a, t, \cdot)$ is constant on E, then $\mu(f, a, t, u) = \|f + a\psi\|$.*

Proof. Let t, u and E be as above. Since E is ever-branching, we can choose elements $u_\sigma \in E$, $\sigma \in \bigcup_{n=0}^{\infty}\{0,1\}^n$, such that $u_\varnothing = u$ and for each σ, $u_\sigma \prec u_{\sigma^\frown 0}, u_{\sigma^\frown 1}$ and $u_{\sigma^\frown 0}$ and $u_{\sigma^\frown 1}$ are incomparable. In this way we construct a dyadic tree inside E of height ω. Define

$$\psi = \mathbf{1}_{(t,u]} + \sum_{n=0}^{\infty} 2^{-n-1} \sum_{\sigma \in \{0,1\}^n} \left(\mathbf{1}_{(u_\sigma, u_{\sigma^\frown 0}]} + \mathbf{1}_{(u_\sigma, u_{\sigma^\frown 1}]}\right).$$

Clearly $\operatorname{supp} \psi \subset (t, u] \cup (u, \infty)$, $\|\psi\|_\infty = 1$ and property (1) above is seen to be satisfied.

Now let $\mu(f, a, t, \cdot)$ be constant on E. For every σ, we have some φ_σ supported on (u_σ, ∞), such that

$$\|f + a\mathbf{1}_{(t,u_\sigma]} + \varphi_\sigma\| - 2^{-n} \leqslant \mu(f, a, t, u_\sigma) = \mu(f, a, t, u).$$

Define $y_n = 2^{-n} \sum_{\sigma \in \{0,1\}^n} (a\mathbf{1}_{(t,u_\sigma]} + \varphi_\sigma)$. Since

$$2^{-n} \sum_{\sigma \in \{0,1\}^n} (f + a\mathbf{1}_{(t,u_\sigma]} + \varphi_\sigma) = f + y_n$$

we obtain

$$\|f + y_n\| \leqslant 2^{-n} \sum_{\sigma \in \{0,1\}^n} \left(\mu(f, a, t, u) + 2^{-n}\right) = \mu(f, a, t, u) + 2^{-n}.$$

Since $\mu(\cdot, a, t, u)$ is Lipschitz, all that is necessary to complete the proof is to show that $\lim \|a\psi - y_n\| = 0$. Observe that the support of $a\psi - y_n$ is contained in the disjoint union of (u_σ, ∞) for $\sigma \in \{0,1\}^n$. Now $\|\psi\restriction_{(u_\sigma,\infty)}\|_\infty \leqslant 2^{-n}$ for $\sigma \in \{0,1\}^n$ and

$$\|\varphi_\sigma\| \leqslant \mu(f, a, t, u) + 2^{-n} + \|f\| + |a|.$$

Therefore $\|a\psi - y_n\|_\infty \leqslant 2^{-n}a + 2^{-n}M(\mu(f, a, t, u) + 2^{-n} + \|f\| + |a|) \to 0$ as required. $\square$

At this point, it is convenient to give a definition and make a couple of remarks.

Definition 19. Given an increasing function $\rho : \mathcal{T} \to \mathbb{R}$, we say that t is a *fan point* of ρ if there exists an ever-branching subset E containing t, such that $\rho(E) = \{\rho(t)\}$. We will call such ψ of Lemma 18 *fan functions*.

Remark 20. Note that infimum attainment in the definition of the μ-functions will be satisfied, regardless of the choice of dyadic tree in Lemma 18.

Remark 21. Suppose that $\rho = \sum_{i \in I} \mu_i$ is a valid sum of μ-functions on $\mathcal{T}$ (where each μ-function is declared to vanish outside its domain). Let t, u, E and ψ be as in Lemma 18. Suppose further that ρ is constant on E. Then every μ_i whose domain includes t is constant on E, and the infimum in the definition of μ_i is attained using ψ, independently of i. Likewise, if t is a bad point for ρ then it is also a bad point for μ_i whenever t is in the domain of μ_i, thus we have infimum attainment for all such i.

Now we can state the result we referred to at the beginning of the section.

Theorem 22. *(See [15, Theorem 8.1].) The space $\mathcal{C}_0(\mathcal{T})$ admits a Talagrand operator if and only if there exists an increasing function $\rho : \mathcal{T} \to \mathbb{R}$ that has no bad points or fan points.*

If we suppose that $\mathcal{T}$ is a tree, such that $\mathcal{C}_0(\mathcal{T})$ admits no Talagrand operators, then every increasing function $\rho : \mathcal{T} \to \mathbb{R}$ has either a bad point or a fan point. Thus, to obtain a proof of Theorem 14, because of Lemma 3, it is enough to prove the following lemma.

Lemma 23. *Suppose that $\mathcal{T}$ is a tree, such that every increasing, real-valued function defined on it has either a bad point or a fan point, and that $\|\cdot\|$ is an equivalent norm on $\mathcal{C}_0(\mathcal{T})$. Then there exists a separable subspace Y of $(\mathcal{C}_0(\mathcal{T}), \|\cdot\|)$ and a sequence of linear operators $\mathcal{L} = \{T_n\colon n \in \mathbb{N}\}$ acting on Y, such that $\|y\| = \sup\{\|T_n y\|\colon n \in \mathbb{N}\}$ for all $y \in Y$ and Y^* is not the norm closed linear span of $\bigcup_{n \in \mathbb{N}} T_n^* Y^*$. As a consequence of Lemma* 3, *Y and $(\mathcal{C}_0(\mathcal{T}), \|\cdot\|)$ are not polyhedral.*

Note that if $\mathcal{T}$ is a tree, such that every increasing function $\rho : \mathcal{T} \to \mathbb{R}$ has either a bad point or a fan point, then one of the following is true: either every increasing function has a bad point, or every such function has a fan point. Indeed, otherwise we could find increasing functions ρ_1 and ρ_2 with no bad points and no fan points respectively. Summing them would produce an increasing function with neither.

So it is possible to split the proof into two cases, the first in which every increasing function has a bad point, and the second in which every such function has a fan point. This we do, due to the unpalatable number of technicalities that arise when trying to tackle both cases simultaneously. It so happens that the machinery required for the bad point case is largely a simplification of that needed for the more complicated fan point case. Thus, to avoid an overly long treatment, we proceed with the fan point case first, and sketch the bad point case afterwards, taking into account all the important differences.

Our present undertaking builds upon a construction that features in [15, Theorem 8.1]. We require some more notation regarding fan points and fan functions. Let us suppose, for the moment, that we have constructed increasing, real-valued functions $\rho_1, \ldots, \rho_n$ on $\mathcal{T}$. Let F_n be the set of fan points of ρ_n and, following Haydon [15, Theorem 8.1], define

$$\varepsilon_{n,t} = \begin{cases} \rho_n(t) & \text{if } t \text{ is a minimal element of } F_n, \\ \rho_n(t) - \sup\{\rho_n(s)\colon s \in (0,t) \cap F_n\} & \text{otherwise.} \end{cases}$$

We let Γ_n be the set of $t \in F_n$ such that $\varepsilon_{n,t} > 0$. Observe the next important fact: if $t \in F_n \setminus \Gamma_n$ then

$$\rho_n(t) = \sup\bigl\{\rho_n(s)\colon s \in (0,t) \cap \Gamma_n\bigr\}. \tag{3.3}$$

Now we depart from Haydon. Given $t \in F_n$, let $u \in [t, \infty)$ be minimal, subject to the requirement that there exist distinct $v_0, v_1 \in u^+$, and ever-branching subsets $E_\theta \in [v_\theta, \infty)$ such that $\rho(E_\theta) = \{\rho_n(v_\theta)\}$, for $\theta \in \{0,1\}$. Notice that u is unique, and so we can define $\pi_n(t)$ to be this u. In addition, we set $\pi_{n,\theta}(t) = v_\theta$ for $\theta \in \{0,1\}$, although these v_θ need not be unique. Now fix fan functions $\psi_{n,t,\theta}$, furnished by Lemma 18, with $\pi_{n,\theta}(t)$ in place of u. In particular, $\psi_{n,t,\theta}$ is supported on $(t, \pi_{n,\theta}(t)] \cup (\pi_{n,\theta}(t), \infty)$ and $\psi_{n,t,\theta}(s) = 1$ whenever $s \in (t, \pi_{n,\theta}(t)]$. We permit a mild abuse of notation while defining $\pi_{n,t}$ and $\psi_{n,t,u}$ by

$$\pi_{n,t}(u) = \begin{cases} \pi_{n,1}(t) & \text{if } \pi_{n,0}(t) \preccurlyeq u, \\ \pi_{n,0}(t) & \text{otherwise,} \end{cases} \quad \text{and} \quad \psi_{n,t,u} = \begin{cases} \psi_{n,t,1} & \text{if } \pi_{n,0}(t) \preccurlyeq u, \\ \psi_{n,t,0} & \text{otherwise.} \end{cases}$$

Next, we define some subspaces of $\mathcal{C}(\mathcal{T})$. Let $t \in \mathcal{T}$ and let $C_{n,t}$ be the closed linear span of

$$\{\mathbf{1}_{(0,t]}\} \cup \bigcup_{k<n} \bigl\{\mathbf{1}_{(s,t]}\colon s \in (0,t) \cap \Gamma_k\bigr\} \cup \bigl\{\psi_{k,s,t}\colon s \in (0,t) \cap \Gamma_k\bigr\}.$$

Since $\sum_{s \in (0,t) \cap \Gamma_n} \varepsilon_{n,s} \leqslant \rho_n(t)$, we see that $(0,t) \cap \Gamma_n$ is countable, hence each $C_{n,t}$ is separable. Let $\{q_l\}_{l=1}^\infty$ be an enumeration of the non-zero rationals and, for each $t \in F_n$, fix a dense sequence $(f_{n,t,k})_{k=1}^\infty$ in $C_{n,t}$ and define the function $\sigma_{n,t} : \mathcal{T} \to \mathbb{R}$ by

$$\sigma_{n,t}(u) = \begin{cases} \sum_{k,l,m=1}^\infty 2^{-k-l-m} \frac{\mu(f_{n,t,k} + q_l \psi_{n,t,u}, q_m, t, u)}{\|f_{n,t,k}\| + |q_l| + |q_m|} & \text{if } u \in (t,\infty) \setminus (t, \pi_n(t)], \\ 0 \quad \text{otherwise.} \end{cases}$$

Because

$$\mu(f + a\psi_{n,t,u}, b, t, u) \leqslant \|f + a\psi_{n,t,u} + b\mathbf{1}_{(t,u]}\| \leqslant \|f\| + |a| + |b|$$

we have $\sigma_{n,t}(u) \leqslant 1$ for all u. Also, $\sigma_{n,t}$ is increasing, for if $u \in (t,\infty) \setminus (t, \pi_n(t)]$ and $u \preccurlyeq v$ then $v \in (t,\infty) \setminus (t, \pi_n(t)]$ and thus $\psi_{n,t,u} = \psi_{n,t,v}$. Now we simply use the fact that $\mu(f, a, t, \cdot)$ is increasing.

Proof of Lemma 23. As previously mentioned, we proceed with the fan point case first. Actually we construct an isomorphic embedding $T : c \oplus c_0 \to \mathcal{C}_0(\mathcal{T})$. After this construction we consider $Y = (T(c \oplus c_0), \|\cdot\|)$ and define a sequence of maps T_n over the isometric copy $X = (c \oplus c_0, |||\cdot|||)$ of Y, where $|||(x,y)||| = \|T(x,y)\|$.

We begin by defining a sequence of increasing functions $\{\rho_n\}_{n=1}^\infty$. We would like this sequence to satisfy the following properties:

(1) $\rho_n(t) \leqslant \rho_{n+1}(t)$ for all t and $\rho_n(u) - \rho_n(t) \leqslant \rho_{n+1}(u) - \rho_{n+1}(t)$ whenever $t \preccurlyeq u$;
(2) $\rho_n(t) \in (0, 2 - 2^{-n})$;
(3) if $k < n$, $t \in \Gamma_k$, $t \preccurlyeq u \preccurlyeq \pi_k(t)$ and $v \in u^+ \setminus (t, \pi_k(t)]$ then $\rho_n(v) > \rho_n(u)$.

We begin by setting $\rho_1(t) = \mu(\mathbf{1}_{(0,t]}, t)$. We have that $\frac{1}{M} \leqslant \rho_1(t) \leqslant 1$, so (2) is clear. Now assume that ρ_n has been constructed. We define

$$\rho_{n+1} = \rho_n + 2^{-n-2} \sum_{t \in \Gamma_n} \varepsilon_{n,t} \sigma_{n,t}$$

and note that this sum is well defined, for

$$\begin{aligned} \rho_{n+1}(u) &= \rho_n(u) + 2^{-n-2} \sum_{t \in (0,u) \cap \Gamma_n} \varepsilon_{n,t} \sigma_{n,t}(u) \\ &\leqslant \rho_n(u) + 2^{-n-2} \sum_{t \in (0,u) \cap \Gamma_n} \varepsilon_{n,t} \\ &\leqslant \left(1 + 2^{-n-2}\right)\rho_n(u) \leqslant \left(1 + 2^{-n-2}\right)\left(2 - 2^{-n}\right) < 2 - 2^{-n-1}. \end{aligned}$$

So we have (2). Since the second summand in the definition of ρ_{n+1} is an increasing, non-negative function, we have (1). Therefore, for (3) to be fulfilled, we simply need to check that $\rho_{n+1}(v) > \rho_{n+1}(u)$ whenever $t \in \Gamma_n$, $t \preccurlyeq u \preccurlyeq \pi_n(t)$ and $v \in u^+ \setminus (t, \pi_n(t)]$. This is indeed so, for $\sigma_{n,t}(v) > 0 = \sigma_{n,t}(u)$. This completes the construction of (ρ_n).

From (1) and (2) we can define $\rho(t) = \sup_n \rho_n(t)$. Since ρ is increasing, it has a fan point w by hypothesis. Note that, again from (1), $\rho_n(u) - \rho_n(t) \leqslant \rho(u) - \rho(t)$ for all n and whenever $t \preccurlyeq u$, thus $w \in F_n$ for all n. By (3), $w \notin \Gamma_n$ for all n. In fact, if $t \in \Gamma_n$ and $t \preccurlyeq u \preccurlyeq \pi_n(t)$, then $u \notin F_{n+1}$. Indeed, if $u \preccurlyeq v$ and $\rho_{n+1}(v) = \rho_{n+1}(u)$ then, necessarily, $v \preccurlyeq \pi_n(t)$ by (3). So the set of $v \in [u, \infty)$ such that $\rho_{n+1}(v) = \rho_{n+1}(u)$ is contained in $[u, \pi_n(t)]$, which is totally ordered and certainly not ever-branching.

If we define the set $\Gamma = \bigcup_n \Gamma_n$, it follows that $\Gamma \cap (0, w)$ countable, because, as we have seen, $\sum_{s \in (0,w) \cap \Gamma_n} \varepsilon_{n,s} \leqslant \rho(w)$ for each n. Let $v = \sup \Gamma \cap (0, w) \preccurlyeq w$. Our intention is to construct strictly increasing sequences $\{n_i\}_{i=1}^{\infty} \subset \mathbb{N}$ and $\{t_i\}_{i=1}^{\infty} \subset \mathcal{T}$ such that $t_i \in \Gamma_{n_i} \cap (0, v)$, $\pi_{n_i}(t_i) \wedge w \prec t_{i+1}$ and $t_i \to v$. Firstly, we show that

$$\text{if } t \in \Gamma_n \cap (0, w) \text{ then there exists } u \in \Gamma_{n+1} \cap \left(\pi_n(t) \wedge w, w\right). \tag{3.4}$$

We can see that $w \not\preccurlyeq \pi_n(t)$. Indeed, $w \in F_{n+1}$, so it cannot be that $w \preccurlyeq \pi_n(t)$ by the corollary of (3) presented above. Let $t' = \pi_n(t) \wedge w \in [t, w)$; also by (3), if t'' is the unique element of $(t')^+ \cap (0, w]$ then $t'' \notin (t, \pi_n(t)]$ by maximality of t' and so $\rho_{n+1}(t') < \rho_{n+1}(t'') \leqslant \rho_{n+1}(w)$. Therefore, by (3.3), there exists $u \in \Gamma_{n+1} \cap (t', w)$ as required.

In particular, $\Gamma \cap (0, w)$ has no greatest element; thus, as $\Gamma \cap (0, w)$ is also countable, we can fix a strictly increasing sequence $\{s_i\}_{i=1}^{\infty} \subset \Gamma \cap (0, v)$, such that $s_i \to v$. Since $w \notin \Gamma_1$, we can find $t_1 \in \Gamma_{n_1} \cap (0, v)$ by (3.3), where $n_1 = 1$. Assume that we have constructed $t_1 \prec t_2 \prec \cdots \prec t_i \prec v$ and corresponding $n_1 < n_2 < \cdots < n_i$, such that $t_j \in \Gamma_{n_j} \cap (0, w)$ for $j \leqslant i$ and $s_j \prec t_{j+1}$ for $j < i$. If $t = \max\{s_i, t_i\} \in \Gamma_n$ then, by repeating (3.4) enough times, we can find $n_{i+1} > n_i, n$ and $t_{i+1} \in \Gamma_{n_{i+1}} \cap (\pi_{n_i}(t_i) \wedge w, w)$.

Since $t_{i+1} \in (t_i, \infty) \setminus (t_i, \pi_{n_i}(t_i)]$ and $t_{i+1} \prec w$, we have $\pi_{n_i,t_i}(t_{i+1}) = \pi_{n_i,t_i}(w)$ and we define u_i to be this element of the tree. By construction, we have ensured that $u_i \not\preccurlyeq t_{i+1}$. It follows that the sets

$$H_i = (t_i, u_i] \cup (u_i, \infty),$$

$i \geqslant 1$, are pairwise disjoint. If we define $\psi_i = \psi_{n_i,t_i,t_{i+1}} = \psi_{n_i,t_i,w}$ then the support of ψ_i lies entirely in H_i, and moreover $\psi_i(u_i) = 1$. Since w is a fan point of ρ, we can fix a fan function ψ, supported on (w, ∞).

Now let us define a linear map $T : c \oplus c_0 \to \mathcal{C}_0(\mathcal{T})$ by

$$T(x, y) = \sum_{i=1}^{\infty} (x_i - x_{i-1})(\mathbf{1}_{(t_{i-1},w]} + \psi) + \sum_{i=1}^{\infty} y_i \psi_i,$$

where $x = \{x_i\}_{i=1}^{\infty} \in c$, $y = \{y_i\}_{i=1}^{\infty} \in c_0$, $x_0 = 0$ and $t_0 = 0$. The second sum in the definition of T is well defined because the supports of ψ_i are pairwise disjoint and $y \in c_0$. The support of $T(x, y)$ is included in the disjoint union of $(0, t_1]$, $(t_i, t_{i+1}] \cup H_i$, $i \geqslant 1$ and $[w, \infty)$. Hence, we see that

$$\begin{aligned}
\|T(x, y)\|_\infty &= \max\Big\{|x_1|, \sup_{i\geqslant 1} \|x_i \mathbf{1}_{(t_i,t_{i+1}]} + y_i \psi_i\|_\infty, |\lim x_i|\Big\} \\
&\leqslant \sup_{i\geqslant 1} |x_i| + |y_i| \\
&\leqslant 2\|(x, y)\|_\infty.
\end{aligned}$$

On the other hand,

$$\|T(x, y)\|_\infty \geqslant |T(x, y)(t_n)| = \Big|\sum_{i\leqslant n} (x_i - x_{i-1})\mathbf{1}_{(t_{i-1},w]}(t_n)\Big| = |x_n|$$

and

$$\|T(x, y)\|_\infty \geqslant |T(x, y)(u_n)| = |y_n \psi_n(u_n)| = |y_n|$$

thus $\|T(x, y)\|_\infty \geqslant \|(x, y)\|_\infty$.

So we have that T is an isomorphic embedding. If $|||(x, y)||| = \|T(x, y)\|$ then $|||\cdot|||$ is equivalent to $\|\cdot\|_\infty$ on $c \oplus c_0$. Put $X = (c \oplus c_0, |||\cdot|||)$ and $Y = (T(c \oplus c_0), \|\cdot\|)$.

Now define maps $P_n, Q_n : c \oplus c_0 \to c_0$ by

$$P_n(x, y)_i = \begin{cases} x_i & \text{if } i \leqslant n, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$Q_n(x, y)_i = \begin{cases} y_i & \text{if } i < n, \\ x_n & \text{if } i = n, \\ 0 & \text{otherwise,} \end{cases}$$

and set $T_n = P_n \oplus Q_n : X \to X$. It is obvious that

$$\ker T_n = \{(x, y) \in X\colon\ x_i = 0 \text{ for } i \leqslant n \text{ and } y_i = 0 \text{ for } i < n\}$$

and if $\xi \in X^*$ is defined by $\xi(x, y) = \lim x_i$ then $\xi \in X^* \setminus \operatorname{cl}\bigcup_n (\ker T_n)^\perp$. Thus the conclusion of Lemma 3 is not satisfied. It remains to show that condition (1.1) of this lemma can be met, and then we will have that X, and thus Y and $(\mathcal{C}_0(\mathcal{T}), \|\cdot\|)$, are not polyhedral.

Let $(x, y) \in c \oplus c_0$ and set $g = T(x, y)$ and $g_i = TT_i(x, y)$. Moreover, set $h = g\mathbf{1}_{\mathcal{T}\setminus(w,\infty)}$ and $h_i = g\mathbf{1}_{\mathcal{T}\setminus(t_i,\infty)}$. For clarity, we note the identities

$$g = \sum_{i=1}^{\infty}(x_i - x_{i-1})\mathbf{1}_{(t_{i-1},w]} + \sum_{i=1}^{\infty} y_i\psi_i + (\lim x_i)\psi = h + (\lim x_i)\psi = h + h(w)\psi$$

and

$$g_i = \sum_{j=1}^{i} x_j\mathbf{1}_{(t_{j-1},t_j]} + \sum_{j=1}^{i-1} y_j\psi_j + x_i\psi_i = h_i + x_i\psi_i = h_i + h_i(t_i)\psi_i.$$

Condition (1.1) of Lemma 3 will be met if we can show that $\|g\| = \sup_i \|g_i\|$. This will follow from the following claims:

(a) $\|g_i\| = \mu(h_i, t_i)$ for all i and $\|g\| = \mu(h, w)$;
(b) $\mu(h, w) = \sup_i \mu(h_i, t_i)$.

We prove claim (a) first. Let C_i be the linear span of

$$\{\mathbf{1}_{(t_j,t_i]}\colon\ 0 \leqslant j < i\} \cup \{\psi_j\colon\ 1 \leqslant j < i\}.$$

Evidently, $\{C_i\}_{i=1}^{\infty}$ forms an increasing sequence of subspaces and $h_i \in C_i$. Note also that since $t_j \in \Gamma_{n_j}$ and $n_j < n_i$ for $j < i$, we have $C_i \subset C_{n_i,t_i}$. Take $i \geqslant 1$. If $t_{i+1} \in F_{n_{i+1}}$ then $t_{i+1} \in F_{n_i+1}$ also, so by construction of ρ_{n_i+1} we have that t_{i+1} is a fan point for σ_{n_i,t_i}. Since we have ensured that $t_{i+1} \in (t_i, \infty) \setminus (t_i, \pi_{n_i}(t_i)]$, it follows that t_{i+1} is a fan point for $\mu(f_{n_i,t_i,k} + q_l\psi_i, q_m, t_i, \cdot)$ for every k, l and m. Therefore

$$\mu(f_{n_i,t_i,k} + q_l\psi_i, q_m, t_i, t_{i+1}) = \|f_{n_i,t_i,k} + q_l\psi_i + q_m(\mathbf{1}_{(t_i,t_{i+1}]} + \psi_{i+1})\|$$

for every k, l and m, by Remark 21. By uniform approximation, it follows that

$$\mu(f + a\psi_i, b, t_i, t_{i+1}) = \|f + a\psi_i + b(\mathbf{1}_{(t_i,t_{i+1}]} + \psi_{i+1})\|$$

for every $f \in C_{n_i,t_i}$ and $a, b \in \mathbb{R}$; equivalently,

$$\begin{aligned}\mu(f + a\psi_i + b\mathbf{1}_{(t_i,t_{i+1}]}, t_{i+1}) &= \mu(f + a\psi_i + b\mathbf{1}_{(t_i,t_{i+1}]}, b, t_{i+1}, t_{i+1}) \\ &= \|(f + a\psi_i + b\mathbf{1}_{(t_i,t_{i+1}]}) + b\psi_{i+1}\|.\end{aligned}$$

The space of functions $f + a\psi_i + b\mathbf{1}_{(t_i,t_{i+1}]}$ of the above form includes C_{i+1}. Thus we have

$$\mu(h_{i+1}, t_{i+1}) = \left\| h_{i+1} + h_{i+1}(t_{i+1})\psi_{i+1} \right\| = \|g_{i+1}\|.$$

It remains to show that $\mu(h_1, t_1) = \|g_1\|$, but this is straightforward, because C_1 is the linear span of $\{\mathbf{1}_{(0,t_1]}\}$ and t_1 is a fan point for $\rho_1 : t \mapsto \mu(\mathbf{1}_{(0,t]}, t)$, so by homogeneity

$$\mu(b\mathbf{1}_{(0,t_1]}, t_1) = \left\| b(\mathbf{1}_{(0,t_1]} + \psi_1) \right\|.$$

Now we show that $\|g\| = \mu(h, w)$. By repeating the above argument with w and ψ in place of t_{i+1} and ψ_{i+1}, respectively, we obtain

$$\begin{aligned} \mu(f + a\psi_i + b\mathbf{1}_{(t_i,w]}, w) &= \mu(f + a\psi_i + b\mathbf{1}_{(t_i,w]}, b, w, w) \\ &= \left\| f + a\psi_i + b(\mathbf{1}_{(t_i,w]} + \psi) \right\| \end{aligned}$$

for every i. Set

$$k_i = h_i + h_i(t_i)\mathbf{1}_{(t_i,w]} = \sum_{j=1}^{i} (x_j - x_{j-1})\mathbf{1}_{(t_{j-1},w]} + \sum_{j=1}^{i-1} y_j\psi_j.$$

From above, we have

$$\mu(k_i, w) = \left\| k_i + h_i(t_i)\psi \right\|$$

and since

$$\|k_i - h\|_\infty = \left\| k_i + h_i(t_i)\psi - g \right\|_\infty \to 0$$

we have $\mu(h, w) = \|g\|$ by another uniform approximation argument. This completes the proof of claim (a).

Now to prove claim (b). First, note that by the remarks about μ-functions made after Definition 15, the sequence $(\mu(h_i, t_i))$ is increasing and bounded above by $\mu(h, w)$. Because $w \in F_n \setminus \Gamma_n$ for all n, we have

$$\rho_n(w) = \sup\left\{ \rho_n(t) \colon t \in (0, w) \cap \Gamma_n \right\}$$

for all n by (3.3). Notice that, if $\rho = \sum_{i\in I} \sigma_i$ is a valid sum of increasing functions and $\rho(u) = \sup\{\rho(t) \colon t \in A\}$, where $A \subset (0, u]$, then $\sigma_i(u) = \sup\{\sigma_i(t) \colon t \in A\}$ for all $i \in I$. Therefore,

$$\sigma_{n_i,t_i}(w) = \left\{ \sigma_{n_i,t_i}(t) \colon t \in (0, w) \cap \Gamma_{n_i} \right\}$$

for all i. By a further uniform approximation argument, we have

$$\mu(f + a\psi_i, b, t_i, w) = \sup\left\{ \mu(f + a\psi_i, b, t_i, t) \colon t \in \left(\pi_{n_i}(t_i) \wedge w, w\right) \cap \Gamma_{n_i} \right\}$$

for all $f \in C_{n_i,t_i}$, $a, b \in \mathbb{R}$ and $i \geqslant 1$. Take $\varepsilon > 0$ and let i be such that, for $j \geqslant i$, we have $\|k_j - k_i\|_\infty \leqslant \varepsilon$. In particular $\|h - k_i\|_\infty \leqslant \varepsilon$, so by the 1-Lipschitz property of μ-functions we have

$$\bigl|\mu(h, w) - \mu(k_i, w)\bigr| \leqslant \varepsilon. \tag{3.5}$$

From above

$$\begin{aligned}\mu(k_i, w) &= \mu\bigl(h_i, h_i(t_i), t_i, w\bigr)\\ &= \sup\bigl\{\mu\bigl(h_i, h_i(t_i), t_i, t\bigr)\colon t \in \bigl(\pi_{n_i}(t_i) \wedge w, w\bigr) \cap \Gamma_{n_i}\bigr\}\end{aligned}$$

thus there exists $j \geqslant i$ such that

$$\bigl|\mu(k_i, w) - \mu\bigl(h_i, h_i(t_i), t_i, t_j\bigr)\bigr| \leqslant \varepsilon \tag{3.6}$$

because $\mu(h_i, h_i(t_i), t_i, \cdot)$ is increasing and $t_j \to v$. Now

$$\mu\bigl(h_i, h_i(t_i), t_i, t_j\bigr) = \mu\bigl(h_i + h_i(t_i)\mathbf{1}_{(t_i,t_j]}, t_j\bigr)$$

so since $\|h_j - (h_i + h_i(t_i)\mathbf{1}_{(t_i,t_j]})\|_\infty = \|k_j - k_i\|_\infty \leqslant \varepsilon$, again by the 1-Lipschitz property, we have therefore

$$\bigl|\mu(h_j, t_j) - \mu\bigl(h_i, h_i(t_i), t_i, t_j\bigr)\bigr| \leqslant \bigl\|h_j - \bigl(h_i + h_i(t_i)\mathbf{1}_{(t_i,t_j]}\bigr)\bigr\| \leqslant \varepsilon \tag{3.7}$$

and putting inequalities (3.5), (3.6) and (3.7) together gives

$$\bigl|\mu(h, w) - \mu(h_j, t_j)\bigr| \leqslant 3\varepsilon$$

which proves claim (b) and completes the proof of the fan point case.

It remains to sketch a proof of the bad point case. It is a simplification of the material above because fan functions cease to matter. In fact, the construction of ρ largely becomes that which is presented in [15, Theorem 8.1]. As above, let us assume the existence of increasing functions $\rho_1, \ldots, \rho_n$, and let B_n denote the set of bad points of ρ_n. Define $\varepsilon_{n,t}$ and Γ_n, with B_n in place of F_n. All π-functions and ψ-functions should be ignored. Thus $C_{n,t}$ becomes the closed linear span of

$$\{\mathbf{1}_{(0,t]}\} \cup \bigcup_{k<n} \bigl\{\mathbf{1}_{(s,t]}\colon s \in (0,t) \cap \Gamma_k\bigr\}$$

and $\sigma_{n,t}$ reduces to

$$\sigma_{n,t}(u) = \begin{cases} \sum_{k,l} 2^{-k-l-2} \frac{\mu(f_{n,t,k}, q_l, t, u)}{\|f_{n,t,k}\| + |q_l|} & \text{if } u \in (t, \infty),\\ 0 \quad \text{otherwise.} \end{cases}$$

The definition of the sequence $\{\rho_n\}_{n=1}^\infty$ remains the same, as do conditions (1) and (2). However condition (3) must be replaced by

(3′) if $k < n$ and $t \in \Gamma_k$ then $\inf_{u \in t^+} \rho_n(u) - \rho_n(t) > 0$.

Arguing as above, for (3′) to be fulfilled, we just need $\inf_{u\in t^+} \rho_{n+1}(u) - \rho_{n+1}(t) > 0$ whenever $t \in \Gamma_n$. This is indeed so, because, for $f \in C_{n,t}$, $a \in \mathbb{R}$ and $t \preccurlyeq u$

$$\mu(f, a, t, u) \geqslant M^{-1}\|f + a\mathbf{1}_{(t,u]}\|_\infty \geqslant (2M)^{-1}\big(\|f\|_\infty + |a|\big)$$

because all elements of this version of $C_{n,t}$ have support disjoint from $a\mathbf{1}_{(t,u]}$. Therefore, for $t \prec u$, $\sigma_{n,t}(u) \geqslant 1/2M > 0 = \sigma_{n,t}(t)$ and so $\rho_{n+1}(u) - \rho_{n+1}(t) \geqslant 2^{n-3}\varepsilon_{n,t}/M$.

By hypothesis, let w be a bad point of ρ, defined as before. Then $w \in B_n$ for all n and, by (3′), $w \notin \Gamma_n$ for any n. Setting $v = \sup \Gamma \cap (0, w)$ as above, we construct strictly increasing sequences $\{n_i\}_{i=1}^\infty \subset \mathbb{N}$ and $\{t_i\}_{i=1}^\infty \subset \mathcal{T}$ such that $t_i \in \Gamma_{n_i} \cap (0, v)$ and $t_i \to v$. We can simplify (3.4) to (3.8) thus:

$$\text{if } t \in \Gamma_n \cap (0, w) \text{ then there exists } u \in \Gamma_{n+1} \cap (t, w). \tag{3.8}$$

Indeed, by (3′), $\rho_{n+1}(t) < \rho_{n+1}(t') \leqslant \rho_{n+1}(w)$, where t' is the unique element of $(0, w] \cap t^+$. By (3.3), we must have some $u \in \Gamma_{n+1} \cap (t, w)$.

Once the sequences have been constructed, we move straight to the definition of $T : (c, \|\cdot\|_\infty) \to (\mathcal{C}_0(\mathcal{T}), \|\cdot\|_\infty)$. Let

$$T(x) = \sum_{i=1}^{\infty} (x_i - x_{i-1})\mathbf{1}_{(t_{i-1},w]}.$$

In this case, T is an isometry. Consider $X = c$ with the norm $|||x||| = \|T(x)\|$, $Y = (T(c), \|\cdot\|)$, and define the maps $T_n : X \to X$ by

$$T_n(x)_i = \begin{cases} x_i & \text{if } i \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that the conclusion of Lemma 3 is not satisfied. To show that condition (1.1) of this lemma holds and conclude that X and thus $(\mathcal{C}_0(\mathcal{T}), \|\cdot\|)$ is not polyhedral, we show that claims

(a′) $\|g_i\| = \mu(g_i, t_i)$ for all i and $\|g\| = \mu(g, w)$;
(b′) $\mu(g, w) = \sup_i \mu(g_i, t_i)$

hold, where g and g_i are as above. We set all fan functions ψ and ψ_i to zero, so that $h = g$ and $h_i = g_i$, and replace all mention of 'fan point' with 'bad point.' We leave it to the reader to verify that the above argument holds with these changes in place. □

4. Non-separable spaces with unconditional basis

In 1994, D. Leung [18] proved that a Banach space X with a shrinking basis $\{e_n\}_{n=1}^\infty$ is isomorphically polyhedral if and only if, there exists an equivalent norm $|||\cdot|||$ on X which is monotone with respect to $\{e_n\}_{n=1}^\infty$ and, for every $x = \sum a_n e_n \in X$, we may find $m \in \mathbb{N}$ such that

$$|||x||| = \left|\left|\left|\sum_{n=1}^{m} a_n e_n\right|\right|\right|.$$

Actually, the sufficient condition of the former result goes back to [3,8]. We extend the above result to Banach spaces with an *uncountable* unconditional basis. In this case we need essential modifications.

Theorem 24. *Let X be a Banach space with a monotone unconditional basis $\{e_i\}_{i\in I}$, i.e. $\|P_\sigma\| = 1$, for any $\sigma \subset I$, $|\sigma| < \infty$, and $P_\sigma x = \sum_{i\in\sigma} e_i^*(x)e_i$, where $\{e_i^*\}_{i\in I}$ is the biorthogonal system for $\{e_i\}_{i\in I}$. Assume that for any $x \in X$ there is $\sigma \subset I$, $|\sigma| < \infty$, with $\|x\| = \|P_\sigma x\|$. Then X is isomorphic to a polyhedral space.*

Proof. Fix a decreasing sequence $\{\varepsilon_k\}_{k=1}^\infty$ of positive numbers with

$$\lim_k \varepsilon_k = 0, \quad 0 < \varepsilon_k < 1/4 \quad \text{and} \quad \varepsilon_{k+1} < \varepsilon_k^2/4. \tag{4.9}$$

From (4.9) we easily get

$$(1 + 4\varepsilon_k)(1 - \varepsilon_k)^2 > 1 + \varepsilon_k^2 > 1 + 4\varepsilon_{k+1}. \tag{4.10}$$

Let $\sigma \subset I$, $|\sigma| = n < \infty$, $L_\sigma = [e_i^*]_{i\in\sigma}$. Let D_σ be a symmetric finite ε_n-net in S_{L_σ}. Put

$$D = \bigcup_{\sigma\subset I,\, |\sigma|<\infty} (1 + 4\varepsilon_{|\sigma|})D_\sigma, \qquad V^* = \text{w}^*\text{-cl}\,\text{co}\, D.$$

Introduce in X a new norm as follows:

$$|||x||| = \max_{f\in V^*} f(x)$$

for $x \in X$. It is not difficult to see that the norm $|||\cdot|||$ is equivalent to the original one, and that V^* is the dual ball in the norm $|||\cdot|||$.

Claim 1.

(a) $|||x||| > \|x\|$, *for any* $x \in X$, $x \neq 0$.
(b) $(1 + \varepsilon_n^2)B_{L_\sigma} \subset (1 - \varepsilon_n)\,\text{co}((1 + 4\varepsilon_n)D_\sigma)$.

Proof. Let $x \in X$, $x \neq 0$, and $\sigma \subset I$, $|\sigma| = n < \infty$, be such that $\|x\| = \|P_\sigma x\|$, and $f \in S_{X^*}$ be such that $f(P_\sigma x) = \|P_\sigma x\|$. If $g = P_\sigma^* f$ then $g(x) = f(P_\sigma x) = \|P_\sigma x\| = \|x\|$. In particular, $g \in S_{L_\sigma}$, and hence there is $h \in D_\sigma$ with $\|g - h\| < \varepsilon_n$. We have (by using (4.10))

$$\begin{aligned}(1 - \varepsilon_n)|||x||| &\geqslant (1 - \varepsilon_n) \max_{f\in(1+4\varepsilon_n)D_\sigma} f(x) \geqslant (1 - \varepsilon_n)(1 + 4\varepsilon_n)h(x) \\ &\geqslant (1 - \varepsilon_n)(1 + 4\varepsilon_n)\big(g(x) - \|g - h\| \times \|x\|\big) \\ &\geqslant (1 + 4\varepsilon_n)(1 - \varepsilon_n)^2\|x\| > \big(1 + \varepsilon_n^2\big)\|x\|,\end{aligned}$$

which finishes the proof of (a). To prove (b) we use the following part of the inequality above

$$(1 - \varepsilon_n) \max_{f\in(1+4\varepsilon_n)D_\sigma} f(x) \geqslant \big(1 + \varepsilon_n^2\big)\|x\|,$$

and the separation theorem. □

Claim 2. *Let σ and ν be two finite subsets of I with $n = |\sigma| < |\nu| = m$. Then*

$$P_\sigma^*\big((1+4\varepsilon_m)D_\nu\big) \subset (1-\varepsilon_n)\operatorname{co}\big((1+4\varepsilon_n)D_\sigma\big).$$

Proof. By using $\|P_\sigma^*\| = 1$, $\varepsilon_m \leqslant \varepsilon_{n+1}$, (4.10) and Claim 1(b), we get

$$P_\sigma^*\big((1+4\varepsilon_m)D_\nu\big) \subset (1+4\varepsilon_{n+1})B_{L_\sigma} \subset \big(1+\varepsilon_n^2\big)B_{L_\sigma} \subset (1-\varepsilon_n)\operatorname{co}\big((1+4\varepsilon_n)D_\sigma\big),$$

finishing the proof. □

Claim 3. *The set D has property* $(*)$ *in the norm* $|||\cdot|||$.

Proof. Let f_0 be a w*-limit point of the set D, i.e. any w*-neighborhood of f_0 contains infinitely many points of D. We will prove that either $|||f_0||| < 1$ or, if $|||f_0||| = 1$, there is not $x \in X$, $|||x||| = 1$, with $f_0(x) = 1$. Put

$$\sigma_0 = \big\{i \in I\colon\ f_0(e_i) \neq 0\big\}, \quad p_0 = |\sigma_0|,$$

and consider two cases.

Case 1. *There is an integer M and a* w**-neighborhood W_0 of f_0 such that for any* w**-neighborhood W of f_0 with $W \subset W_0$, and for any $g_\sigma \in (1+4\varepsilon_{|\sigma|})D_\sigma$ with $g_\sigma \in W$, we have $|\sigma| \leqslant M$.*

In this case we can assume without loss of generality that there is an integer $p \leqslant M$ and that f_0 is a w*-limit point of a net $\{g_\sigma\}$, with $|\sigma| = p$ for any σ. It is not difficult to see that $p_0 \leqslant p$. Moreover since f_0 is a w*-limit point of D and each D_σ is finite, it easily follows that $p_0 < p$. Now from Claim 2 we get $|||f_0||| < 1$.

Case 2. *For any integer M and for any* w**-neighborhood W_0 of f_0 there is a* w**-neighborhood W of f_0 with $W \subset W_0$, such that there is $g_\sigma \in (1+4\varepsilon_{|\sigma|})D_\sigma$ with $g_\sigma \in W$, with $|\sigma| > M$.*

In this case $f_0 \in B_{X^*}$. Assume that $|||f_0||| = 1$ and there is $x \in X$, $|||x||| = 1$, with $f_0(x) = 1$. Since $f_0 \in B_{X^*}$ it follows that $\|x\| \geqslant 1$. However from Claim 1 we get $|||x||| > \|x\|$, for any $x \in X$, $x \neq 0$, which is a contradiction. □

The theorem now follows from Claim 3 and Proposition 7. □

Recall that a non-degenerate Orlicz function M is a non-decreasing convex function defined on $t \geqslant 0$, with $M(0) = 0$, $M(t) > 0$ for all $t > 0$, and $\lim_{t\to+\infty} M(t) = +\infty$. For a set Γ, the Orlicz space $\ell_M(\Gamma)$ consists of all real functions x defined on Γ such that $\sum_{\gamma\in\Gamma} M(|x(\gamma)|/\rho) < +\infty$ for some $\rho > 0$, equipped with the norm

$$\|x\| = \inf\Big\{\rho > 0\colon \sum_{\gamma\in\Gamma} M\big(\big|x(\gamma)\big|/\rho\big) \leqslant 1\Big\}.$$

Let $\{e_\gamma\}_{\gamma\in\Gamma}$ be the family of the functions $e_\gamma(\beta)=\delta_{\gamma,\beta}$, whose sole non-zero value is 1 at $\beta=\gamma$, and $h_M(\Gamma)$ be the closed subspace of $\ell_M(\Gamma)$ generated by this family. Clearly $\{e_\gamma\}_{\gamma\in\Gamma}$ is a monotone, symmetric basis of the Banach space $h_M(\Gamma)$.

Corollary 25. *Let M be a non-degenerate Orlicz function such that there exists a finite number K satisfying*

$$\lim_{t\to 0}\frac{M(Kt)}{M(t)}=+\infty.$$

Then the spaces $h_M(\Gamma)$ admits a polyhedral renorming, for any Γ.

Proof. The construction of the desired norm runs along the lines of the construction in the proof of Theorem 4 from [18]. To prove that this norm is polyhedral we use Theorem 24. □

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